Announcements

1) Problem #3 on HW 1 has been changed,

d) on #5 moved to HW 2

Note: ($\mathbb{C}$ and Δ inequality)

In $\mathbb{C}$, the triangle inequality still holds with

$$|x+iy| = \sqrt{x^2+y^2},$$

you just need to work harder to prove it.

Definition: (Vector Space)

Let $\mathbb{F}$ be a field.

A **vector space over $\mathbb{F}$**

is a set V and

two binary operations

"$+$": $V \times V \to V$

(vector addition)

"$\cdot$": $\mathbb{F} \times V \to V$

(scalar multiplication)

Such that

1) $(V, +)$ is an abelian group

2) (distributivity of scalar multiplication)

$\forall \alpha, \beta \in \mathbb{F}$ and all $u, w \in V$,

$$\boxed{(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u}$$

and

$$\boxed{\alpha \cdot (u + w) = \alpha \cdot u + \alpha \cdot w}$$

3) (associativity of scalar multiplication)

$\forall\ \alpha, \beta \in \mathbb{F}$ and $w \in V$,

$$\boxed{\alpha \cdot (\beta \cdot w) = (\alpha\beta) \cdot w}$$

4) If $1_{\mathbb{F}}$ is the multiplicative identity of $\mathbb{F}$ and all w in V,

$$\boxed{1_{\mathbb{F}} \cdot w = w.}$$

Example 1: $(\mathbb{F}^n)$

Let $\mathbb{F}$ be any field.

$\mathbb{F}^n = V$ is the vector space of all ordered n-tuples of elements of $\mathbb{F}$.

$n=1$: $\mathbb{F}$

$n=2$: (α_1, α_2), $\alpha_1, \alpha_2 \in \mathbb{F}$

$n=3$: $(\alpha_1, \alpha_2, \alpha_3)$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$

etc.

If $w=(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{F}^n$

$u=(\beta_1, \beta_2, \cdots, \beta_n) \in \mathbb{F}^n$

and $\alpha \in \mathbb{F}$,

define

$$\boxed{u+w=(\alpha_1+\beta_1, \alpha_2+\beta_2, \cdots, \alpha_n+\beta_n)}$$

and

$$\boxed{\alpha \cdot w=(\alpha\alpha_1, \alpha\alpha_2, \cdots, \alpha\alpha_n)}$$

By definition, $+ : V \times V \to V$

and $\cdot : \mathbb{F} \times V \to V$

since adding or multiplying two elements of $\mathbb{F}$ yields another element of $\mathbb{F}$.

Check this is a vector space!

1) $(\mathbb{F}^n, +)$ is an abelian group

associativity follows from associativity of $\mathbb{F}$ (done component-wise)

Identity

Identity of $\mathbb{F}^n$:

If $0_{\mathbb{F}}$ is the additive identity of $\mathbb{F}$, the identity element of $\mathbb{F}^n$ is

$$0_{\mathbb{F}^n} = (\underbrace{0_{\mathbb{F}}, 0_{\mathbb{F}}, \dots, 0_{\mathbb{F}}}_{n \text{ times}})$$

Check:

$$0_{\mathbb{F}^n} + w$$

$$= (0_{\mathbb{F}} + \alpha_1, 0_{\mathbb{F}} + \alpha_2, \ldots, 0_{\mathbb{F}} + \alpha_n)$$

$$= (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$= w$$

$$= (\alpha_1 + 0_{\mathbb{F}}, \alpha_2 + 0_{\mathbb{F}}, \ldots, \alpha_n + 0_{\mathbb{F}})$$

$$= w + 0_{\mathbb{F}^n}$$ ✓

Inverses:

The inverse of w is

$$-w = (-\alpha_1, -\alpha_2, \ldots, -\alpha_n)$$

Check:

$$w + (-w) = (\alpha_1 - \alpha_1, \alpha_2 - \alpha_2, \ldots, \alpha_n - \alpha_n)$$

$$= (0_{\mathbb{F}}, 0_{\mathbb{F}}, \ldots, 0_{\mathbb{F}})$$

$$= (-\alpha_1 + \alpha_1, -\alpha_2 + \alpha_2, \ldots, -\alpha_n + \alpha_n)$$

$$= (-w) + w$$

Commutativity

$$u+w=(\beta_1+\alpha_1, \beta_2+\alpha_2, \ldots, \beta_n+\alpha_n)$$

$$=(\alpha_1+\beta_1, \alpha_2+\beta_2, \ldots, \alpha_n+\beta_n)$$

Since addition in $\mathbb{F}$ is commutative

$$=w+u$$

2) Distributivity

Let $\alpha, \beta \in \mathbb{F}$.

$(\alpha+\beta)\cdot w$

$= ((\alpha+\beta)\alpha_1, (\alpha+\beta)\alpha_2, \dots, (\alpha+\beta)\alpha_n)$

$= (\alpha\alpha_1+\beta\alpha_1, \alpha\alpha_2+\beta\alpha_2, \dots, \alpha\alpha_n+\beta\alpha_n)$

by distributivity in $\mathbb{F}$

$= (\alpha\alpha_1, \alpha\alpha_2, \dots, \alpha\alpha_n) + (\beta\alpha_1, \beta\alpha_2, \dots, \beta\alpha_n)$

$= \alpha\cdot w + \beta\cdot w$ ✓

$$\alpha \cdot (u+w)$$

$$= \alpha \cdot (\alpha_1+\beta_1, \alpha_2+\beta_2, \dots, \alpha_n+\beta_n)$$

$$= (\alpha\,(\alpha_1+\beta_1), \alpha(\alpha_2+\beta_2), \dots, \alpha(\alpha_n+\beta_n))$$

$$= (\alpha\alpha_1+\alpha\beta_1, \alpha\alpha_2+\alpha\beta_2, \dots, \alpha\alpha_n+\alpha\beta_n)$$

by distributivity in $\mathbb{F}$

$$= (\alpha\alpha_1, \alpha\alpha_2, \dots, \alpha\alpha_n) + (\alpha\beta_1, \alpha\beta_2, \dots, \alpha\beta_n)$$

$$= \alpha \cdot u + \alpha w$$

3) Associativity

$$(\alpha\beta)\cdot w$$

$$= ((\alpha\beta)\alpha_1, (\alpha\beta)\alpha_2, \ldots, (\alpha\beta)\alpha_n)$$

$$= (\alpha(\beta\alpha_1), \alpha(\beta\alpha_2), \ldots, \alpha(\beta\alpha_n))$$

by associativity in $\mathbb{F}$

$$= \alpha\cdot(\beta\alpha_1, \beta\alpha_2, \ldots, \beta\alpha_n)$$

$$= \alpha\cdot(\beta\cdot w)$$ ✓

4) Let $1_{\mathbb{F}}$ be the field identity for multiplication

$$1_{\mathbb{F}} \cdot w = (1_{\mathbb{F}}\alpha_1, 1_{\mathbb{F}}\alpha_2, \ldots, 1_{\mathbb{F}}\alpha_n)$$

$$= (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$= w$$

So $\mathbb{F}^n$ is a vector space over $\mathbb{F}$!

Subsequent examples will omit proofs.

Example 2: $(M_n(\mathbb{F}))$

$n \times n$ matrices over $\mathbb{F}$

$(\alpha_{i,j})_{i,j=1}^{n}$ with $\alpha_{i,j} \in \mathbb{F}$

for all $1 \le i,j \le n$ is

a vector space over $\mathbb{F}$.

If $(\beta_{i,j})_{i,j=1}^{n} \in M_n(\mathbb{F})$

and $\alpha \in \mathbb{F}$ define

$$(\alpha_{i,j})_{i,j=1}^{n} + (\beta_{i,j})_{i,j=1}^{n}$$

$$= (\alpha_{i,j} + \beta_{i,j})_{i,j=1}^{n}$$

(addition)

$$\alpha(\alpha_{i,j})_{i,j=1}^{n}$$

$$= (\alpha\alpha_{i,j})_{i,j=1}^{n}$$

(scalar multiplication)

Example 3: ($\mathbb{C}$ and $\mathbb{R}$)

$\mathbb{C}$ is a vector space over $\mathbb{R}$.

Vector addition is just addition in $\mathbb{C}$ and scalar multiplication is multiplication in $\mathbb{C}$ restricted to $\mathbb{R} \times \mathbb{C}$.

Example 4 (sequence space)

Let $\mathbb{F}$ be any field and let V be the set of all sequences of elements of $\mathbb{F}$, indexed by $\mathbb{N}$.

Then V is a vector space with the following operations:

If $(\alpha_i)_{i=1}^{\infty}, (\beta_i)_{i=1}^{\infty} \in V$

and $\alpha \in \mathbb{F}$,

define

$$(\alpha_i)_{i=1}^{\infty} + (\beta_i)_{i=1}^{\infty} = (\alpha_i + \beta_i)_{i=1}^{\infty}$$

(addition)

$$\alpha \cdot (\alpha_i)_{i=1}^{\infty} = (\alpha \alpha_i)_{i=1}^{\infty}$$

(scalar multiplication)

Example 5: Let $\mathbb{F}$ be a field and let V be all functions from $\mathbb{F}$ to $\mathbb{F}$.

Then V is a vector space, with the following operations:

If $f, g : \mathbb{F} \to \mathbb{F}$

and $\alpha, \beta \in \mathbb{F}$, define

$$(f+g)(\beta) = f(\beta) + g(\beta)$$

(addition)

$$(\alpha \cdot f)(\beta) = \alpha \cdot f(\beta)$$

(scalar multiplication)